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Determinant representation for a quantum correlation function of the lattice sine–Gordon model

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Abstract. We consider a completely integrable lattice regularization of the sine–Gordon model with discrete space and continuous time. We derive a determinant representation for a correlation function which in the continuum limit turns into the correlation function of local fields. The determinant is then embedded into a system of integrable integro-differential equations. The leading asymptotic behaviour of the correlation function is described in terms of the solution of a Riemann–Hilbert Problem (RHP) related to the system of integro-differential equations. The leading term in the asymptotical decomposition of the solution of the RHP is obtained.

1. Introduction

The sine–Gordon model is completely integrable (exactly solvable) both on the classical and on the quantum level [1–7]. We shall write the sine–Gordon equation in the following form:

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + \frac{m^2}{\beta}\sin\beta u(x,t) = 0. \tag{1.1}$$

Here m is a mass, β is the coupling constant. For later convenience we also introduce

$$\gamma = \frac{\beta^2}{8}.$$

In the classical case $u(x,t)$ is a function of two variables, x and t are space and time coordinates. In the quantum case $u(x,t)$ is a local quantum field of the sine–Gordon model. The Hamiltonian reads

$$\mathcal{H} = \int \mathrm{d}x\ \left(\frac{1}{2}p^2 + \frac{1}{2}(\partial_x u)^2 + \frac{m^2}{\beta^2}(1-\cos\ \beta u)\right). \tag{1.2}$$

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Momentum and topological charge are given by

$$P = -\int \mathrm{d}x\; p\partial_x u \qquad Q = \frac{\beta}{2\pi}\int \mathrm{d}x\; \partial_x u. \tag{1.3}$$

Here $p(x,t) = \partial_t u(x,t)$ and $u(x,t)$ satisfy Poisson brackets $\{p(x), u(y)\} = \delta(x-y)$. Equation (1.1) has a Lax representation and a classical r-matrix [1–6]. After quantization, the fields u and p satisfy canonical commutation relations $[u(x), p(y)] = \mathrm{i}\delta(x-y)$. The physical ground state $|\Omega\rangle$ of the quantum system can be obtained by filling the Dirac sea of negative energy pseudoparticles [7].

Let us now consider the quantum operator

$$\exp(\alpha Q(x)) = \exp\left\{\frac{\alpha\beta}{2\pi}(u(x) - u(0))\right\} \qquad Q(x) = \frac{\beta}{2\pi}\int_0^x \mathrm{d}z\; \partial_z u(z) \tag{1.4}$$

where $Q(x)$ measures the topological charge on the interval $[0, x]$. In this paper we show how to represent the correlation function

$$\langle\Omega|\exp(\alpha Q(x))|\Omega\rangle \tag{1.5}$$

as the determinant of an integral operator (in fact we shall see below, that the coefficient α in (1.5) needs to be renormalized). Note that via differentiation with respect to α we can obtain correlation functions of local quantum fields from (1.5). We shall consider the quantum version of (1.2) in the region $\frac{\pi}{2} < \gamma < \frac{2\pi}{3}$ (many of our intermediate results hold in larger regions of coupling constant). Note that $\gamma \to 0$ is the quasiclassical region of the sine–Gordon model and at $\gamma = \pi/2$ the spectrum of the Hamiltonian is equivalent to free fermions. To deal with the ultraviolet divergences of the continuum model we shall employ a suitably chosen lattice regularization.

The determinant representation then permits us to describe the correlation functions in terms of a system of integrable integro-differential equations. These equations can be solved by means of a Riemann–Hilbert problem (RHP) which in turn enables one to obtain elementary formulae for the asymptotics of the correlation functions. This program has first been applied to the nonlinear Schrödinger equation in [8] and is described in detail in the book [9] (see also [10]).

There has been previous work on determining correlation functions in the sine–Gordon model. Form factors were determined by Smirnov in [11, 12]. At the free fermionic point $\gamma = \pi/2$ a determinant representation of the correlation function (1.5) has been constructed using the coordinate Bethe ansatz in [13]. A description of a different correlator at the free fermionic point through a Fredholm determinant (derived from a form factor sum) which in turn satisfies an integrable differential (sinh-Gordon) equation has been obtained in [14]. In this paper we start the investigation of correlation functions in the sine–Gordon model for general γ, in particular away from the free fermionic point in the framework of its solution [15, 16] by means of the quantum inverse scattering method (QISM).

The plan of this paper is as follows: in section 2 we review the integrable lattice regularization of the sine–Gordon model introduced in [16]. The algebraic Bethe ansatz is formulated and the construction of the ground state [17] is discussed. In section 3 we derive the determinant representation of the correlator (1.5) for the range of coupling constants stated above. As this part of the analysis is very similar to the analogous problem for the spin-$\frac{1}{2}$ Heisenberg XXZ model (which was treated in full detail in [22]) we omit many details and only give an account of the main steps without providing proofs (which can be found in [22]). In sections 4 and 5 we embed the determinant representation into a system of integrable integro-differential equations and in section 6 the related RHP is formulated and the leading asymptotic behaviour of the correlation function is extracted.

2. Lattice sine–Gordon

2.1. $\mathcal{L}$-operator

We shall consider a lattice version of the sine–Gordon model which is also completely integrable. It will have exactly the same r-matrix (both in the classical and quantum case) as the continuous model. The elementary $\mathcal{L}$-operator of the LSG model is [15, 16]

$$\mathcal{L}(n|\lambda) = \begin{pmatrix} e^{-i\beta p_n/8}\rho_n e^{-i\beta p_n/8} & \frac{1}{2}m\Delta\sinh(\lambda - i\beta u_n/2) \\ -\frac{1}{2}m\Delta\sinh(\lambda + i\beta u_n/2) & e^{i\beta p_n/8}\rho_n e^{i\beta p_n/8} \end{pmatrix} \tag{2.1}$$

Here Δ is the lattice constant and p_n, u_n are the dynamical variables on site n of the lattice. In the quantum model they obey canonical commutation relations $[u_n, p_m] = i\delta_{nm}$. Furthermore, we have introduced

$$\rho_n = (1 + 2S\cos\beta u_n)^{\frac{1}{2}} \qquad S = (\tfrac{1}{4}m\Delta)^2. \tag{2.2}$$

The symmetries of the $\mathcal{L}$-operator of the LSG model are expressed by the identities (the asterisk means Hermitian conjugation of the quantum operators)

$$\sigma^y\mathcal{L}^*(n|\bar{\lambda})\sigma^y = \mathcal{L}(n|\lambda) \qquad \sigma^z\mathcal{L}(n|\lambda)\sigma^z = \mathcal{L}(n|\lambda + i\pi). \tag{2.3}$$

Its quantum determinant [15, 16] is

$$\det_q \mathcal{L}(n|\lambda) \equiv 1 + 2S\cosh 2\lambda. \tag{2.4}$$

The $\mathcal{L}$-operator (2.1) satisfies the Yang–Baxter equation

$$R(\lambda,\mu)(\mathcal{L}(n|\lambda)\otimes\mathcal{L}(n|\mu)) = (\mathcal{L}(n|\mu)\otimes\mathcal{L}(n|\lambda))R(\lambda,\mu). \tag{2.5}$$

$R(\lambda,\mu)$ in equation (2.5) is the standard sine–Gordon R-matrix given by the following expression:

$$R(\lambda,\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}. \tag{2.6}$$

Here

$$f(\mu,\lambda) = \frac{\sinh(\mu - \lambda - i\gamma)}{\sinh(\mu - \lambda)} \qquad g(\mu,\lambda) = -i\frac{\sin\gamma}{\sinh(\mu - \lambda)}. \tag{2.7}$$

In different sites of the lattice the matrix elements of $\mathcal{L}$ commute. As usual in the QISM we define the monodromy matrix by taking products of the $\mathcal{L}$-operators in matrix space:

$$\mathcal{T}(\lambda) = \mathcal{L}(L|\lambda)\mathcal{L}(L-1|\lambda)\ldots\mathcal{L}(1|\lambda) \tag{2.8}$$

where L is the number of sites in the lattice which we take to be even. By construction this operator also satisfies a Yang–Baxter equation

$$R(\lambda,\mu)(\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)) = (\mathcal{T}(\mu)\otimes\mathcal{T}(\lambda))R(\lambda,\mu). \tag{2.9}$$

It might be interesting to point out that the entries fo the $\mathcal{L}$-operator (2.1) form a representation of a quantum group: The operators (we suppress the site index n)

$$S^+ = \frac{2}{i\sin\gamma\ m\Delta}e^{i\beta p/8}\rho e^{i\beta p/8}$$

$$S^- = \frac{-2}{i\sin\gamma\ m\Delta}e^{-i\beta p/8}\rho e^{-i\beta p/8}$$

$$S^0 = e^{-i\beta u/2} \qquad S^1 = e^{i\beta u/2}$$

satisfy the commutation relations of the quadratic (Sklyanin) algebra

$$[S^+, S^-] = \frac{1}{q - q^{-1}}((S^0)^2 - (S^1)^2) \qquad [S^0, S^1] = 0$$
$$S^\pm\ S^0 = q^{\mp 1} S^0 S^\pm \qquad S^\pm\ S^1 = q^{\pm 1} S^1 S^\pm$$

with $q = \exp(\mathrm{i}\gamma)$. For q being a root of unity this algebra has finite dimensional cyclic representations: for rational values of the parameter $\gamma/\pi = Q/P$ the quantum operators entering the $\mathcal{L}$-operator can be written as $2P \times 2P$ matrices with elements

$$\chi = \mathrm{e}^{\mathrm{i}\beta u/2} \to \delta_{ab}\mathrm{e}^{\mathrm{i}\pi(a-1)/P} \qquad \pi = \mathrm{e}^{\mathrm{i}\beta p/4} \to \delta_{a+Q,b} \qquad a, b = 1, \ldots, 2P,\ \ a + 2P \equiv a.$$

The definition of the $\mathcal{L}$ operator alone does not determine a definite lattice model: in addition the Hamiltonian of the lattice sine–Gordon model needs to be specified. For this choice there exist several different possibilities (see [15, 16, 18]). All of them are completely integrable and can in fact be diagonalized simultaneously. Furthermore, all of them have the same continuum limit (1.2). They differ from one another by higher orders in the lattice spacing Δ. While all of them can be considered equivalently as a lattice regularization of the continuum model we shall show below, how a *unique* lattice Hamiltonian can be chosen by requiring that it has the 'same' ground state wavefunction as the continuum model. This choice of the Hamiltonian will bring the dynamics of the lattice model as close as possible to that of the continuum model.

2.2. Algebraic Bethe ansatz for the lattice sine–Gordon model

We shall consider the monodromy matrix (2.8)

$$\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2.10}$$

As a direct consequence of the Yang–Baxter equation (2.5) for $\mathcal{T}(\lambda)$ the trace of the monodromy matrix, the so-called transfer matrix

$$\tau(\lambda) = \mathrm{trace}\,\mathcal{T}(\lambda) = A(\lambda) + D(\lambda) \tag{2.11}$$

commutes for different values of the spectral parameter λ, i.e. $[\tau(\lambda), \tau(\mu)] = 0$. Hence, it is the generator of commuting integrals for the system which are diagonalized by the algebraic *Bethe ansatz*. The starting point is the 'pseudo-vacuum' (or reference state). To construct this simple eigenstate of $\tau(\lambda)$ we combine the $\mathcal{L}$-operators in pairs:

$$\hat{\mathcal{L}}(n|\lambda) = \mathcal{L}(2n|\lambda)\mathcal{L}(2n-1|\lambda) \equiv \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & \delta_n(\lambda) \end{pmatrix}. \tag{2.12}$$

Choosing

$$\langle u|0\rangle_n = \left\{1 - 2S\cos\frac{\beta}{2}(u_{2n} + u_{2n-1})\right\}^{-\frac{1}{2}} \delta\left(u_{2n} - u_{2n-1} - \frac{\beta}{4} + \frac{2\pi}{\beta}\right) \tag{2.13}$$

(for rational $\gamma/\pi = Q/P$ the δ-function can be replaced by a Kronecker δ-symbol and $|0\rangle_n$ will become normalizable) we find from (2.12)

$$\begin{aligned} &\gamma_n(\lambda)|0\rangle_n = 0 \\ &\alpha_n(\lambda)|0\rangle_n = \{1 + 2S\cosh(2\lambda - \mathrm{i}\gamma)\}|0\rangle_n \\ &\delta_n(\lambda)|0\rangle_n = \{1 + 2S\cosh(2\lambda + \mathrm{i}\gamma)\}|0\rangle_n. \end{aligned} \tag{2.14}$$

Now we can follow the standard steps of the algebraic Bethe ansatz. As a consequence of (2.14) the 'global pseudo-vacuum'

$$|0\rangle = \prod_{n=1}^{L/2} |0\rangle_n \tag{2.15}$$

is an eigenstate of the operators $A(\lambda)$ and $D(\lambda)$ (and hence the transfer matrix (2.11)) with eigenvalues $a(\lambda)$ and $d(\lambda)$, respectively:

$$a(\lambda) = \{1 + 2S\cosh(2\lambda - \mathrm{i}\gamma)\}^{\frac{L}{2}} \qquad d(\lambda) = \{1 + 2S\cosh(2\lambda + \mathrm{i}\gamma)\}^{\frac{L}{2}}. \tag{2.16}$$

More eigenfunctions of the transfer matrix are found by acting with the operator $B(\lambda)$ on the pseudo-vacuum

$$\prod_{j=1}^{N} B(\lambda_j)|0\rangle \tag{2.17}$$

provided that the $\{\lambda_j\}$ satisfy the Bethe ansatz equations

$$\left(\frac{1 + 2S\cosh(2\lambda_j - \mathrm{i}\gamma)}{1 + 2S\cosh(2\lambda_j + \mathrm{i}\gamma)}\right)^{\frac{L}{2}} = -\prod_{k=1}^{N} \frac{\sinh(\lambda_j - \lambda_k + \mathrm{i}\gamma)}{\sinh(\lambda_j - \lambda_k - \mathrm{i}\gamma)}. \tag{2.18}$$

The corresponding eigenvalue of the transfer matrix (2.11) is

$$\Lambda(\lambda|\lambda_j) = a(\lambda)\prod_{j=1}^{N} f(\lambda, \lambda_j) + d(\lambda)\prod_{j=1}^{N} f(\lambda_j, \lambda) \tag{2.19}$$

where $f(\lambda, \mu)$ has been defined in (2.7).

The number N of the Bethe ansatz roots λ_j can be identified with the topological charge (1.3). The correct lattice version in the quantum case is

$$Q = \frac{4}{\beta}\sum_{n=1}^{L/2}(u_{2n} - u_{2n-1}) + L\frac{\pi - \gamma}{2\gamma}. \tag{2.20}$$

The difference in the coefficient compared to (1.3) is related to the fractional charge of the excitations. In [19] it was shown that the fractional charge appears due to the repulsion beyond the cut-off in the process of ultraviolet renormalization. Equation (2.20) is the number operator for particles

$$Q\prod_{j=1}^{N} B(\lambda_j)|0\rangle = N\prod_{j=1}^{N} B(\lambda_j)|0\rangle.$$

One can prove that (here σ^z is the Pauli matrix in the matrix space)

$$[Q, \mathcal{T}(\lambda)] = \tfrac{1}{2}[\sigma^z, \mathcal{T}(\lambda)].$$

Now we can discuss our choice of the lattice Hamiltonian for the lattice sine–Gordon model. As mentioned above we want to construct a lattice version resembling the dynamics of the continuum model as closely as possible. According to the standard quantization of the sine–Gordon model the ground state of the continuum model contains no bound states (strings).

For possible lattice models we shall concentrate on the two integrable models introduced in [15, 16, 18]. The latter has been constructed by Tarasov, Takhtajan and Faddeev (TTF) such that it contains interactions of nearest neighbours on the lattice only. The ground state for this Hamiltonian was found in [20]: in addition to a Dirac sea of elementary particles it contains bound states. In the continuum limit the density of the bound states vanishes, thus, reproducing the known results for the continuum model. Apart from the Hamiltonian the QISM yields higher integrals of motion. These describe interactions over larger distances. Adding these interaction terms to the TTF Hamiltonian with coefficients vanishing in the continuum limit $\Delta \to 0$ produces *different* lattice Hamiltonians with the *same* continuum limit while preserving integrability. This is the origin of the freedom in choice of the lattice hamiltonian.

Another Hamiltonian for the lattice sine–Gordon model has been introduced in [15, 16]. The corresponding ground state for this Hamiltonian has been constructed by Bogoliubov [17]: he was able to prove that in the interval $\pi/3 \leqslant \gamma \leqslant 2\pi/3$ the ground state is built from elementary particles only—just as in the continuum model. Furthermore, he found that the set of observable excitations coincides with the continuum model. Hence, unlike the situation in the TTF model no phase transition is met in performing the continuum limit. For the reasons stated above we choose this Hamiltonian for our studies of correlation functions.

It is given in terms of trace identities. Expressing the zeroes $d(\kappa_\pm) = 0$ and $a(\nu_\pm) = 0$ of (2.16) as

$$e^{2\kappa_\pm} = -b^{\pm 1} e^{-i\gamma} \qquad e^{2\nu_\pm} = -b^{\pm 1} e^{i\gamma} \qquad \text{where } b = \frac{2S}{1+\sqrt{1-4S^2}} \tag{2.21}$$

($\lambda_\pm = \frac{1}{2}(i\pi \pm \ln b)$ are the zeroes of the quantum determinant (2.4) of $\mathcal{L}$). The Hamiltonian of the lattice sine–Gordon model considered here is given by

$$\begin{aligned}\mathcal{H}_{LSG} = -\frac{m^2\Delta}{32b\sin\gamma} \Bigg\{ & e^{i\gamma} \left(\frac{\partial}{\partial\lambda} \ln \frac{\tau(\lambda)}{a(\lambda)}\right)_{\lambda=\kappa_+} - e^{-i\gamma} \left(\frac{\partial}{\partial\lambda} \ln \frac{\tau(\lambda)}{a(\lambda)}\right)_{\lambda=\kappa_-} \\ & + e^{-i\gamma} \left(\frac{\partial}{\partial\lambda} \ln \frac{\tau(\lambda)}{d(\lambda)}\right)_{\lambda=\nu_+} - e^{i\gamma} \left(\frac{\partial}{\partial\lambda} \ln \frac{\tau(\lambda)}{d(\lambda)}\right)_{\lambda=\nu_-} \Bigg\}.\end{aligned} \tag{2.22}$$

This is the model studied in [15, 16]. From (2.19) one finds that (2.17) are eigenfunctions of this Hamiltonian with energy eigenvalues given by

$$\mathcal{H}_{LSG} \prod_{j=1}^{N} B(\lambda_j)|0\rangle = \left(\sum_{k=1}^{N} h(\lambda_k)\right) \prod_{j=1}^{N} B(\lambda_j)|0\rangle \tag{2.23}$$

with the single particle energies

$$\begin{aligned} h(\lambda) = \frac{m^2\Delta}{32bi} \Bigg\{ & \frac{e^{i\gamma}}{\sinh(\kappa_+ - \lambda)\sinh(\kappa_+ - \lambda_j - i\gamma)} - \frac{e^{-i\gamma}}{\sinh(\kappa_- - \lambda)\sinh(\kappa_- - \lambda - i\gamma)} \\ & - \frac{e^{-i\gamma}}{\sinh(\nu_+ - \lambda)\sinh(\nu_+ - \lambda + i\gamma)} + \frac{e^{i\gamma}}{\sinh(\nu_- - \lambda_j)\sinh(\nu_- - \lambda + i\gamma)} \Bigg\}. \end{aligned} \tag{2.24}$$

In the continuum limit $\Delta \to 0$ (which is reached by letting $b \to 0$ here) one immediately reproduces the result [7]

$$h(\lambda)|_{\Delta\to 0} = \tfrac{1}{2} m^2 \Delta \sin\gamma \cosh 2\lambda$$

for the single particle dispersion of the continuum model.

To find the solution of (2.18) corresponding to the ground state of the model it is necessary to classify the possible configurations of λ_j in the complex plane according to the so called string hypothesis [21]. The details of this are not important in the present context. It was found by Bogoliubov [17] that the ground state of (2.22) is obtained by filling all permitted states of pseudoparticles with rapidities λ_j on the line $\mathrm{Im}\,\lambda = \pi/2$. Taking the logarithm of equation (2.18) in such a state one obtains

$$\frac{L}{2}\,p\left(\lambda_j + \mathrm{i}\frac{\pi}{2}\right) = 2\pi Q_j - \mathrm{i}\sum_k \ln\left(\frac{\sinh(\lambda_j - \lambda_k + \mathrm{i}\gamma)}{\sinh(\lambda_j - \lambda_k - \mathrm{i}\gamma)}\right). \tag{2.25}$$

Here the Q_j are distinct integers characterizing the state uniquely and

$$p\left(\lambda + \mathrm{i}\frac{\pi}{2}\right) = -\mathrm{i}\ln\left(\frac{1 - 2S\cosh(2\lambda - \mathrm{i}\gamma)}{1 - 2S\cosh(2\lambda + \mathrm{i}\gamma)}\right).$$

In the thermodynamic limit the density $\rho(\lambda_j) = \frac{1}{L}\partial Q_j/\partial\lambda_j$ is then given in terms of the integral equation

$$\frac{1}{2}p'\left(\lambda + \mathrm{i}\frac{\pi}{2}\right) = 2\pi\rho(\lambda) + \int_{-\infty}^{+\infty} \mathrm{d}\mu\, K(\lambda - \mu)\rho(\mu) \tag{2.26}$$

where

$$K(\lambda) = \frac{-\sin 2\gamma}{\sinh(\lambda + \mathrm{i}\gamma)\sinh(\lambda - \mathrm{i}\gamma)} = \frac{-2\sin 2\gamma}{\cosh 2\lambda - \cos 2\gamma}. \tag{2.27}$$

This integral equation can be solved by Fourier transform resulting in ($\Lambda = -\ln b$)

$$\rho(\lambda) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k\lambda}\frac{\sinh\frac{1}{2}k(\pi - \gamma)}{\sinh\frac{1}{2}k\gamma\;\cosh\frac{1}{2}k(\pi - \gamma)}\cos\frac{1}{2}k\Lambda. \tag{2.28}$$

Similarly one can compute the excitation energies. This is useful to find the correct mass renormalization formula. To perform the continuum limit of the sine–Gordon model one should let $\Delta \to 0$ and simultaneously $m \to \infty$ as

$$m = \text{constant}\;\Delta^{-\gamma/\pi}. \tag{2.29}$$

3. Algebraic formulation of correlation functions

For the evaluation of the correlation function (1.5) we shall make extensive use of the similarity (in the framework of the QISM) of the LSG model with the spin-$\frac{1}{2}$ XXZ Heisenberg chain which is derived from a monodromy matrix satisfying a Yang-Baxter equation with the same R-matrix (2.6) as the present model. The correlation functions corresponding to (1.5) in the XXZ model have have recently been studied in [22, 23].

First we note, that the symmetry of the $\mathcal{L}$ operator (2.3) implies for the ground state configuration consisting of rapidities $\{\tilde{\lambda}_j = \lambda_j + \mathrm{i}\pi/2\}$ with real λ_j

$$(B(\tilde{\lambda}_j))^\dagger = C(\tilde{\lambda}_j).$$

In order to express the correlation function (1.5) in the algebraic framework outlined above we first need to define the lattice analogue of the operator $Q(x)$ in (1.4). The correct expression is found to be

$$Q_1(n) = \frac{4}{\beta}\sum_{k=1}^{n/2}(u_{2k} - u_{2k-1}) + n\left(\frac{\pi - \gamma}{2\gamma}\right) \tag{3.1}$$

which counts the number of particles in the interval $[1, n]$ (n even). In the continuum limit this expression becomes

$$Q_1(n) \to \frac{2}{\beta}(u(x) - u(0)) + \frac{x}{\Delta}\left(\frac{\pi - \gamma}{2\gamma}\right) \qquad x = n\Delta. \tag{3.2}$$

Hence the lattice analogue of the correlation function (1.5) can be written as

$$\langle\Omega|\exp(\alpha Q_1(n))|\Omega\rangle \equiv \frac{\langle 0|\prod_{j=1}^N C(\tilde\lambda_j)\exp(\alpha Q_1(n))\prod_{k=1}^N B(\tilde\lambda_k)|0\rangle}{\langle 0|\prod_{j=1}^N C(\tilde\lambda_j)\prod_{k=1}^N B(\tilde\lambda_k)|0\rangle} \tag{3.3}$$

where $\tilde\lambda_j$ are solutions of the Bethe ansatz equations (2.18) for the ground state configuration.

Let us first study the norm appearing in the denominator of this expression. To evaluate this expression one should commute the $C(\tilde\lambda_j)$ to the right of the product where they annihilate the pseudo-vacuum $|0\rangle$. Since the commutation relations between the elements of the monodromy matrix (2.10) are completely determined by the R-matrix we can use the result of [24, 22] (see also [25, 26]) for the norm of Bethe ansatz states (after identifying γ with $2(\pi-\eta)$ in paper [22])

$$\langle 0|\prod_{j=1}^N C(\tilde\lambda_j)\prod_{j=1}^N B(\tilde\lambda_k)|0\rangle = (-\sin\gamma)^N\left\{\prod_{j\neq k} f(\lambda_j,\lambda_k)\right\}\left\{\prod_{j=1}^N a(\tilde\lambda_j)d(\tilde\lambda_j)\right\}\det\mathcal{N} \tag{3.4}$$

where the $N\times N$ matrix $\mathcal{N}$ is given by

$$\mathcal{N}_{jk} = \delta_{jk}\left\{\mathrm{i}\frac{\partial}{\partial\tilde\lambda_j}\ln\frac{a(\tilde\lambda_j)}{d(\tilde\lambda_j)} + \sum_{n=1}^N K(\tilde\lambda_j-\tilde\lambda_n)\right\} - K(\tilde\lambda_j-\tilde\lambda_k).$$

The functions $K(\lambda)$ and $a(\lambda)$, $d(\lambda)$ have been introduced in the previous section. In the thermodynamic limit this expression can be further simplified: We rewrite $\mathcal{N}=\mathcal{I}\cdot\mathcal{J}$ where

$$\mathcal{I}_{jk} = \delta_{jk} - \frac{K(\lambda_j-\lambda_k)}{\theta_k} \qquad \mathcal{J}_{jk} = \delta_{jk}\theta_j$$

$$\theta_j = \mathrm{i}\frac{\partial}{\partial\tilde\lambda_j}\ln\frac{a(\tilde\lambda_j)}{d(\tilde\lambda_j)} + \sum_{n=1}^N K(\tilde\lambda_j-\tilde\lambda_n).$$

Comparing the last expression with equations (2.25) and (2.26) for the ground state density of particles one obtains $\theta_j = -2\pi L\rho(\lambda_j)$. Performing the thermodynamic limit on the matrix $\mathcal{I}$ one finds that it turns into a Fredholm integral operator $\hat{\mathcal{I}} = 1 + \frac{1}{2\pi}\hat{K}$ acting as

$$\hat{\mathcal{I}} * f|_\lambda = f(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\mu\, K(\lambda-\mu)f(\mu). \tag{3.5}$$

Here $K(\lambda)$ is the kernel given in (2.27).

Putting everything together we find

$$\langle 0|\prod_{j=1}^N C(\tilde\lambda_j)\prod_{j=1}^N B(\tilde\lambda_k)|0\rangle = (2\pi L\sin\gamma)^N\left\{\prod_{j\neq k} f(\lambda_j,\lambda_k)\right\}$$
$$\times\left\{\prod_{j=1}^N a(\tilde\lambda_j)d(\tilde\lambda_j)\rho(\lambda_j)\right\}\det\left(1+\frac{1}{2\pi}\hat{K}\right). \tag{3.6}$$

We now turn to the numerator of (3.3): to reduce the evaluation of the expectation value of $\exp(\alpha Q_1(n))$ in a Bethe state (2.17) to the computation of scalar products we divide the lattice of length L into two sub-chains of length n and $L-n$ and associate a monodromy matrix with each of them, namely

$$\mathcal{T}(\lambda) = \mathcal{T}(2,\lambda)\mathcal{T}(1,\lambda) \qquad \mathcal{T}(i,\lambda) = \begin{pmatrix} A_i(\lambda) & B(\lambda_i) \\ C_i(\lambda) & D(\lambda_i)\end{pmatrix} \qquad i=1,2. \tag{3.7}$$

In terms of $\mathcal{L}$-operators they are given by

$$\mathcal{T}(2,\lambda) = \mathcal{L}(L,\lambda)\mathcal{L}(L-1,\lambda)\ldots\mathcal{L}(n+1,\lambda)$$
$$\mathcal{T}(1,\lambda) = \mathcal{L}(n,\lambda)\mathcal{L}(n-1,\lambda)\ldots\mathcal{L}(1,\lambda).$$

By construction these monodromy matrices satisfy the same Yang–Baxter equation (2.9) as $\mathcal{T}(\lambda)$. Similarly, the global reference state (2.15) can be decomposed into a direct product of pseudo vacua for the subchains $|0\rangle_2 \otimes |0\rangle_1$ (remember that we have chosen n to be even) which are eigenstates of $A_i(\lambda)$ and $D_i(\lambda)$

$$A_i(\lambda)|0\rangle_i = a_i(\lambda)|0\rangle_i \qquad D_i(\lambda)|0\rangle_i = d_i(\lambda)|0\rangle_i \tag{3.8}$$

where $a_i(\lambda)$ and $d_i(\lambda)$ are given by (2.16) with L replaced by n and $L-n$ for $i = 1, 2$, respectively. The creation and annihilation operators $B_i(\lambda)$ and $C_i(\lambda)$ act according to

$$C_i(\lambda)|0\rangle_i = 0 \qquad \langle 0|B_i(\lambda) = 0. \tag{3.9}$$

In this decomposed quantum space the numerator of (3.3) can be rewritten as (see e.g. [9, 22])

$$\begin{aligned}\sum {}_1\langle 0| \prod_{I_C} C_1(\tilde{\lambda}^C_{I_C}) \prod_{I_B} B_1(\tilde{\lambda}^B_{I_B})|0\rangle_1 \, {}_2\langle 0| \prod_{II_C} C_2(\tilde{\lambda}^C_{II_C}) \prod_{II_B} B_2(\tilde{\lambda}^B_{II_B})|0\rangle_2 \\ \times e^{\alpha n_1}\left\{ \prod_{I_B, I_C} a_2(\tilde{\lambda}^B_{I_B}) d_2(\tilde{\lambda}^C_{I_C})\right\}\left\{ \prod_{II_B, II_C} a_1(\tilde{\lambda}^C_{II_C}) d_1(\tilde{\lambda}^B_{II_B})\right\} \\ \times\left\{ \prod_{I_B, II_B} f(\lambda^B_{I_B}, \lambda^B_{II_B})\right\}\left\{ \prod_{I_C, II_C} f(\lambda^C_{II_C}, \lambda^C_{I_C})\right\}\end{aligned} \tag{3.10}$$

where the sum is over all partitions

$$\{\tilde{\lambda}^B_{I_B}\} \cup \{\tilde{\lambda}^B_{II_B}\} = \{\tilde{\lambda}\} \qquad \{\tilde{\lambda}^B_{I_B}\} \cap \{\tilde{\lambda}^B_{II_B}\} = \emptyset$$
$$\{\tilde{\lambda}^C_{I_C}\} \cup \{\tilde{\lambda}^C_{II_C}\} = \{\tilde{\lambda}\} \qquad \{\tilde{\lambda}^C_{I_C}\} \cap \{\tilde{\lambda}^C_{II_C}\} = \emptyset$$

of the set $\{\tilde{\lambda}\}$ with card$\{\tilde{\lambda}_{I_B}\}$ = card$\{\tilde{\lambda}_{I_C}\} = n_1$, card$\{\tilde{\lambda}_{II_C}\}$ = card$\{\tilde{\lambda}_{II_B}\} = N - n_1$. Due to (3.9) we only need to consider partitions such that the sizes of I_B and I_C (and II_B and II_C) are the same. We next turn to an investigation of the scalar products occurring in (3.10). Owing to (3.8) and (3.9) and the fact that the monodromy matrices $\mathcal{T}(i, \lambda)$ fulfill the same Yang–Baxter equation (2.9) as $\mathcal{T}(\lambda)$ it is sufficient to consider scalar products on the entire lattice

$$S_N = \langle 0| \prod_{j=1}^{N} C(\lambda^C_j) \prod_{k=1}^{N} B(\lambda^B_k)|0\rangle.$$

Here we do not assume that the sets of spectral parameters $\{\lambda^B\}$ and $\{\lambda^C\}$ are the same, and we also do not impose the Bethe equations (2.18). From (2.9) and the action on the reference state $A(\lambda)|0\rangle = a(\lambda)|0\rangle$, $D(\lambda)|0\rangle = d(\lambda)|0\rangle$ it follows that scalar products can be represented as

$$S_N = \sum_{A,D} \prod_{j=1}^{N} a(\lambda^A_j) \prod_{k=1}^{N} d(\lambda^D_k) K_N \begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^A\} & \{\lambda^D\} \end{pmatrix} \tag{3.11}$$

where the sum is over all partitions of $\{\lambda^C\} \cup \{\lambda^B\}$ into two sets $\{\lambda^A\}$ and $\{\lambda^D\}$. The coefficients K_N are functions of the λ_j and are *completely determined by the intertwining relation* (2.9). The R-matrix (2.6) is, however, identical to the one for the spin-$\frac{1}{2}$ Heisenberg XXZ model (after appropriate identifications of the coupling constants). This implies that the coefficients K_N for the sine–Gordon model and the XXZ chain are identical, so that we can take over the result for the XXZ case (see e.g. [22]). The main point is that the K_N's can be represented as *determinants*. This is done in two steps: first the so-called *highest coefficients*, which are obtained for the partition $\{\lambda^A\} = \{\lambda^C\}$, $\{\lambda^D\} = \{\lambda^B\}$, are represented as determinants

$$K_N\begin{pmatrix}\{\lambda^C\} & \{\lambda^B\}\\ \{\lambda^C\} & \{\lambda^B\}\end{pmatrix} = \left(\prod_{j>k} g(\lambda_j^B,\lambda_k^B)g(\lambda_k^C,\lambda_j^C)\right)\prod_{j,k} h(\lambda_j^C,\lambda_k^B)\det(M_C^B)$$
$$h(\mu,\nu) = \frac{f(\mu,\nu)}{g(\mu,\nu)} \qquad (M_C^B)_{jk} = \frac{g(\lambda_j^C,\lambda_k^B)}{h(\lambda_j^C,\lambda_k^B)} = t(\lambda_j^C,\lambda_k^B) \tag{3.12}$$

where from (2.7)

$$h(\lambda,\mu) = \frac{\sinh(\lambda-\mu-\mathrm{i}\gamma)}{-\mathrm{i}\sin\gamma} \qquad t(\lambda,\mu) = \frac{-\sin^2\gamma}{\sinh(\lambda-\mu-\mathrm{i}\gamma)\sinh(\lambda-\mu)}.$$

In the second step arbitrary coefficients K_N are then expressed in terms of highest coefficients as follows

$$K_N\begin{pmatrix}\{\lambda^C\} & \{\lambda^B\}\\ \{\lambda^A\} & \{\lambda^D\}\end{pmatrix} = \left(\prod_{j\in AC}\prod_{k\in DC} f(\lambda_j^{AC},\lambda_k^{DC})\right)\left(\prod_{l\in AB}\prod_{m\in DB} f(\lambda_l^{AB},\lambda_m^{DB})\right)$$
$$\times K_n\begin{pmatrix}\{\lambda^{AB}\} & \{\lambda^{DC}\}\\ \{\lambda^{AB}\} & \{\lambda^{DC}\}\end{pmatrix} K_{N-n}\begin{pmatrix}\{\lambda^{AC}\} & \{\lambda^{DB}\}\\ \{\lambda^{AC}\} & \{\lambda^{DB}\}\end{pmatrix}. \tag{3.13}$$

Using (3.12) and (3.13) in (3.11) we obtain the following expression for general scalar products in the lattice sine–Gordon model

$$S_N = \prod_{j>k} g(\lambda_j^C,\lambda_k^C)g(\lambda_k^B,\lambda_j^B)\sum \mathrm{sgn}(P_C)\,\mathrm{sgn}(P_B)\prod_{j,k} h(\lambda_j^{AB},\lambda_k^{DC})\prod_{l,m} h(\lambda_l^{AC},\lambda_m^{DB})$$
$$\times\prod_{l,k} h(\lambda_l^{AC},\lambda_k^{DC})\prod_{j,m} h(\lambda_j^{AB},\lambda_m^{DB})\det(M_{DC}^{AB})\det(M_{DB}^{AC}) \tag{3.14}$$

where P_C is the permutation $\{\lambda_1^{AC},\ldots,\lambda_n^{AC},\lambda_1^{DC},\ldots,\lambda_{N-n}^{DC}\}$ of $\{\lambda_1^C,\ldots,\lambda_N^C\}$, P_B is the permutation $\{\lambda_1^{DB},\ldots,\lambda_n^{DB},\lambda_1^{AB},\ldots,\lambda_{N-n}^{AB}\}$ of $\{\lambda_1^B,\ldots,\lambda_N^B\}$, $\mathrm{sgn}(P)$ is the sign of the permutation P, and

$$(M_{DC}^{AB})_{jk} = t(\lambda_j^{AB},\lambda_k^{DC})d(\lambda_k^{DC})a(\lambda_j^{AB}). \tag{3.15}$$

Following the steps first carried out in [27] it is now possible to represent S_N as a single determinant. The discussion for sine–Gordon is identical to the only for the XXZ chain [22] so that we only present a brief discussion of the necessary steps and give the final result. We first note that the sum on the r.h.s. in (3.14) looks very similar to a Laplace decomposition of the determinant of the *sum* of two matrices $(S_1)_{jk} = t(\lambda_j^C,\lambda_k^B)a(\lambda_j^C)d(\lambda_k^B)$ and $(S_2)_{jk} = t(\lambda_k^B,\lambda_j^C)d(\lambda_j^C)a(\lambda_k^B)$ (see e.g. [9] p 221). However, this does not reproduce the $h(\lambda,\mu)$-factors. This leads to the introduction of a *dual quantum field* $\varphi(\lambda)$ acting in a bosonic Fock space with vacua $|0)$ and $(\tilde{0}|$† according to

$$\varphi(\lambda) = p(\lambda) + q(\lambda) \qquad [\varphi(\lambda),\varphi(\mu)] = 0 \qquad (\tilde{0}|q(\lambda) = 0 = p(\lambda)|0)$$
$$[p(\lambda),q(\mu)] = -\ln(h(\lambda,\mu)h(\mu,\lambda)) \qquad [p(\lambda),p(\mu)] = 0 = [q(\lambda),q(\mu)]. \tag{3.16}$$

We emphasize that the field φ commutes for different values of spectral parameters. Using the dual field it is now possible to recast (3.14) as a *single* determinant of the sum of two

† We use the same notation as in [22].

matrices

$$S_N = \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \prod_{j=1}^{N} a(\lambda_j^C) d(\lambda_j^B) \prod_{j,k} h(\lambda_j^C, \lambda_k^B) (\tilde{0}| \det S |0)$$
$$S_{jk} = t(\lambda_j^C, \lambda_k^B) + t(\lambda_k^B, \lambda_j^C) \frac{r(\lambda_k^B)}{r(\lambda_j^C)} \exp(\varphi(\lambda_k^B) - \varphi(\lambda_j^C)) \prod_{m=1}^{N} \frac{h(\lambda_k^B, \lambda_m^B) h(\lambda_m^C, \lambda_j^C)}{h(\lambda_m^C, \lambda_k^B) h(\lambda_j^C, \lambda_m^B)} \tag{3.17}$$

where $r(\lambda) = \frac{a(\lambda)}{d(\lambda)}$. The consequence of representing S_N as a single determinant is the occurrence of the expectation value in the dual space.

Using (3.17) in (3.10) and then applying the dual field trick several times it is possible to represent (3.10) as a single determinant of the sum of four matrices. This analysis is completely analogous to the XXZ case treated in [22] so that we only state the result:

$$\langle 0| \prod_{j=1}^{N} C(\tilde{\lambda}_j) \exp(\alpha Q_1(n)) \prod_{k=1}^{N} B(\tilde{\lambda}_k) |0\rangle = \left\{ \prod_{j \neq k} f(\lambda_j, \lambda_k) \right\} \left\{ \prod_{j=1}^{N} a(\tilde{\lambda}_j) d(\tilde{\lambda}_j) \right\} (\tilde{0}| \det \mathcal{G} |0)$$
$$\begin{aligned} \mathcal{G}_{jk} = {} & t(\tilde{\lambda}_j, \tilde{\lambda}_k) + t(\tilde{\lambda}_k, \tilde{\lambda}_j) \frac{r_1(\tilde{\lambda}_j)}{r_1(\tilde{\lambda}_k)} \exp(\varphi_2(\tilde{\lambda}_k) - \varphi_2(\tilde{\lambda}_j)) + \exp(\alpha + \varphi_4(\tilde{\lambda}_k) - \varphi_3(\tilde{\lambda}_j)) \\ & \times \left[t(\tilde{\lambda}_k, \tilde{\lambda}_j) + t(\tilde{\lambda}_j, \tilde{\lambda}_k) \frac{r_1(\tilde{\lambda}_j)}{r_1(\tilde{\lambda}_k)} \exp(\varphi_1(\tilde{\lambda}_j) - \varphi_1(\tilde{\lambda}_k)) \right] \\ & - \mathrm{i} \delta_{jk} \sin\gamma \frac{\partial}{\partial \tilde{\lambda}_j} \left(\ln(r(\tilde{\lambda}_j)) + \sum_{\substack{n=1 \\ n \neq j}}^{N} \ln \left[\frac{h(\tilde{\lambda}_j, \tilde{\lambda}_n)}{h(\tilde{\lambda}_n, \tilde{\lambda}_j)} \right] \right) \end{aligned} \tag{3.18}$$

where

$$r_1(\lambda) = a_1(\lambda)/d_1(\lambda) = \left(\frac{1 + 2S \cosh(2\lambda - \mathrm{i}\gamma)}{1 + 2S \cosh(2\lambda + \mathrm{i}\gamma)} \right)^{\frac{n}{2}}$$

and the commuting dual fields φ_a are defined according to

$$\varphi_a(\lambda) = p_a(\lambda) + q_a(\lambda) \qquad (\tilde{0}| q_a(\lambda) = 0 = p_a(\lambda)|0) \qquad (\tilde{0}|0) = 1 \qquad a = 1 \ldots 4$$
$$[q_b(\mu), p_a(\lambda)] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \ln(h(\lambda, \mu)) + \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \ln(h(\mu, \lambda)) \tag{3.19}$$

where $a, b = 1 \ldots 4$. Here all terms not proportional to δ_{jk} in $\mathcal{G}_{jk}$ are understood in the sense of l'Hospital for the diagonal elements. In the thermodynamic limit further simplifications take place. Following the analysis for the norms above we express $\mathcal{G}$ as the product of two matrices $\mathcal{J}$ and $\mathcal{W}$

$$\mathcal{G} = -(\sin\gamma) \mathcal{W} \mathcal{J} \qquad \mathcal{J}_{jk} = \delta_{jk} \theta_k \qquad \mathcal{W}_{jk} = \delta_{jk} - \frac{1}{\theta_k} \mathcal{V}(\tilde{\lambda}_j, \tilde{\lambda}_k) \tag{3.20}$$

where $\theta_j = -2\pi L \rho(\lambda_j)$ and

$$\begin{aligned} (\sin\gamma) \mathcal{V}(\lambda, \mu) = {} & t(\lambda, \mu) + t(\mu, \lambda) \frac{r_1(\lambda)}{r_1(\mu)} \exp(\varphi_2(\mu) - \varphi_2(\lambda)) + \exp(\alpha + \varphi_4(\mu) - \varphi_3(\lambda)) \\ & \times \left[t(\mu, \lambda) + t(\lambda, \mu) \frac{r_1(\lambda)}{r_1(\mu)} \exp(\varphi_1(\lambda) - \varphi_1(\mu)) \right]. \end{aligned} \tag{3.21}$$

In the thermodynamic limit $\mathcal{W}$ turns into an integral operator $\hat{\mathcal{W}} = 1 + \frac{1}{2\pi}\hat{V}$ acting as

$$\left(1 + \frac{1}{2\pi}\hat{V}\right) * f|_\lambda = f(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\mu\, V(\lambda,\mu) f(\mu) \tag{3.22}$$

where the integral kernel is obtained from (3.21) as (the arguments of the dual fields are shifted by $\mathrm{i}\pi/2$ which does not alter the defining commutation relations (3.19))

$$V(\lambda,\mu) = \frac{-\sin\gamma}{\sinh(\lambda-\mu)}\left\{\frac{1}{\sinh(\lambda-\mu-\mathrm{i}\gamma)} + \frac{e_2^{-1}(\lambda)e_2(\mu)}{\sinh(\lambda-\mu+\mathrm{i}\gamma)}\right.$$
$$\left. + \exp(\alpha+\varphi_4(\mu)-\varphi_3(\lambda))\left(\frac{1}{\sinh(\lambda-\mu+\mathrm{i}\gamma)} + \frac{e_1^{-1}(\mu)e_1(\lambda)}{\sinh(\lambda-\mu-\mathrm{i}\gamma)}\right)\right\} \tag{3.23}$$

with

$$e_2(\lambda) = \left(\frac{1-2S\cosh(2\lambda+\mathrm{i}\gamma)}{1-2S\cosh(2\lambda-\mathrm{i}\gamma)}\right)^{\frac{n}{2}} \mathrm{e}^{\varphi_2(\lambda)} \qquad e_1(\lambda) = \left(\frac{1-2S\cosh(2\lambda-\mathrm{i}\gamma)}{1-2S\cosh(2\lambda+\mathrm{i}\gamma)}\right)^{\frac{n}{2}} \mathrm{e}^{\varphi_1(\lambda)}.$$

Putting everything together we thus find

$$\langle\Omega|\exp(\alpha Q_1(n))|\Omega\rangle \equiv \frac{(\tilde{0}|\det(1+\frac{1}{2\pi}\hat{V})|0)}{\det(1+\frac{1}{2\pi}\hat{K})} \tag{3.24}$$

where $1+\frac{1}{2\pi}\hat{V}$ and $1+\frac{1}{2\pi}\hat{K}$ are integral operators acting according to (3.22) and (3.5) with kernels defined in (2.27) and (3.23).

4. Continuum limit

As mentioned in the introduction the purpose of the present work is to determine correlators for the SG Quantum Field Theory, and the lattice model studied above is used merely as a regulator for the UV divergences. We are therefore interested in the *continuum limit* of the determinant representation (3.24). As mentioned above the SG Quantum Field Theory is recovered from the lattice regularization by taking the lattice spacing to zero $\Delta \to 0$ and simultaneously the bare mass m to infinity keeping $m\Delta^{\frac{\gamma}{\pi}}$ fixed [17]. In order to take the continuum limit we now employ the following regularization for the integral operators in (3.24): we restrict the integration for the integral operator $1+\frac{1}{2\pi}\hat{V}$ to the interval $[-\Lambda,\Lambda]$, and then take $\Delta \to 0$ in such a way that $S\cosh(2\lambda) \ll 1\ \forall\lambda \in [-\Lambda,\Lambda]$ (recall (2.2) for the relation of S and Δ). Using this regularization the $e_j(\lambda)$'s simplify to

$$e_2(\lambda) = \exp(-\mathrm{i}p\sinh(2\lambda)+\varphi_2(\lambda)) \qquad e_1(\lambda) = \exp(\mathrm{i}p\sinh(2\lambda)+\varphi_1(\lambda)) \tag{4.1}$$

where

$$p = \frac{c^2}{8}\Delta^{\frac{\pi-2\gamma}{\pi}}\sin(\gamma)n\Delta. \tag{4.2}$$

Here we have used (2.29) and $n\Delta = x$ should be identified with the continuum distance. The constant c is given in terms of the physical soliton mass.

This regularization allows to embed the determinant (3.24) into a system of integrable integro-differential equations which we shall need later to determine the subleading terms in the asymptotic expansion of the correlation functions. With (4.1) the kernel (3.23) can be brought into standard form [9]. We perform a change of variables $z = \exp(2\lambda)$, and replace the factors $(\sinh(\lambda-\mu\pm\mathrm{i}\gamma))^{-1}$ in (3.23) by an integration over an exponential.

Then the transpose of the kernel (3.23) reads (up to a similarity transform which leaves the determinant unchanged)

$$\frac{1}{2\pi}V^T(z_1,z_2) = \frac{\mathrm{i}}{z_1 - z_2}\int_0^\infty \mathrm{d}s \sum_{j=1}^{4} E_j(z_2|s)e_j(z_1|s) \tag{4.3}$$

where

$$\begin{aligned}
e_1(z|s) &= \frac{\kappa}{\sqrt{2\pi}}\exp(\varphi_4(z))|2,z,s\rangle \\
E_1(z|s) &= -\frac{\kappa}{\sqrt{2\pi}}\exp(-\varphi_3(z))\langle 2,z,s| \\
e_2(z|s) &= \frac{1}{\sqrt{2\pi}}|1,z,s\rangle \\
E_2(z|s) &= \frac{1}{\sqrt{2\pi}}\langle 1,z,s| \\
e_3(z|s) &= \frac{1}{\sqrt{2\pi}}\exp(-\mathrm{i}pk(z)+\varphi_2(z))|2,z,s\rangle \\
E_3(z|s) &= -\frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}pk(z)-\varphi_2(z))\langle 2,z,s| \\
e_4(z|s) &= \frac{\kappa}{\sqrt{2\pi}}\exp(-\mathrm{i}pk(z)-\varphi_1(z)+\varphi_4(z))|1,z,s\rangle \\
E_4(z|s) &= \frac{\kappa}{\sqrt{2\pi}}\exp(\mathrm{i}pk(z)+\varphi_1(z)-\varphi_3(z))\langle 1,z,s|.
\end{aligned} \tag{4.4}$$

Here we use the notation $k(z) = \frac{1}{2}(z - z^{-1})$, $w = \exp(\mathrm{i}\gamma)$, $\kappa = \exp(\frac{\alpha}{2})$, and

$$\begin{aligned}
|1,z,s\rangle &= \sqrt{2z\sin(\gamma)}\exp(\mathrm{i}zws) = \langle 2,z,s| \\
|2,z,s\rangle &= \sqrt{2z\sin(\gamma)}\exp\left(-\mathrm{i}\frac{z}{w}s\right) = \langle 1,z,s|
\end{aligned} \tag{4.5}$$

are normalized in such a way that $\langle 1|1\rangle = \int_0^\infty \mathrm{d}s\,\langle 1,z,s|1,z,s\rangle = 1$, and similarly $\langle 2|2\rangle = 1$.

The inverse of the integral operator $1 + \frac{1}{2\pi}\hat{V}^T$ is defined by

$$\begin{aligned}
(1-\hat{R}) * \left(1 + \frac{1}{2\pi}\hat{V}^T\right) &= 1 = \left(1 + \frac{1}{2\pi}\hat{V}^T\right) * (1-\hat{R}) \\
\hat{R} &= \left(1 + \frac{1}{2\pi}\hat{V}^T\right)^{-1} * \frac{1}{2\pi}\hat{V}^T.
\end{aligned} \tag{4.6}$$

In terms of the functions $f_j(z|s)$, $F_j(z|s)$

$$(1-\hat{R}) * e_j|_{z,s} = f_j(z|s) \qquad E_j * (1-\hat{R})|_{z,s} = F_j(z|s) \tag{4.7}$$

the kernel of $\hat{R}$ can be written in a form similar to (4.3)

$$R(z_1,z_2) = \frac{\mathrm{i}}{z_1 - z_2}\sum_{j=1}^{4}\int_0^\infty \mathrm{d}s\, f_j(z_1|s)F_j(z_2|s). \tag{4.8}$$

as can be seen by acting with $(1 + \frac{1}{2\pi}\hat{V}^T)$ on (4.8).

5. Integro-differential equations

Let us now derive integro-differential equations (IDE) determining the functions $f_j(z|s)$ and $F_j(z|s)$. The analogue of these equations in the case of impenetrable bosons proved very useful for the anlysis of the corresponding RHP and we expect the equations below to play a similar role for the problem at hand. To this end we consider derivatives with respect to p and the integration boundary Λ. For the Λ-derivatives we find

$$\begin{aligned} &\partial_\Lambda f_j(z|s) + \sum_{l=1}^{4} \int_0^\infty \mathrm{d}t\, U_{jl}(z|s,t) f_l(z|t) = 0 \\ &\partial_\Lambda F_j(z|s) - \sum_{l=1}^{4} \int_0^\infty \mathrm{d}t\, F_l(z|t) U_{lj}(z|t,s) = 0 \end{aligned} \tag{5.1}$$

where

$$U_{jk}(z|s,t) = \frac{2\mathrm{i}\mathrm{e}^{2\Lambda}}{z-\mathrm{e}^{2\Lambda}} f_j(\mathrm{e}^{2\Lambda}|s) F_k(\mathrm{e}^{2\Lambda}|t) + \frac{2\mathrm{i}\mathrm{e}^{-2\Lambda}}{z-\mathrm{e}^{-2\Lambda}} f_j(\mathrm{e}^{-2\Lambda}|s) F_k(\mathrm{e}^{-2\Lambda}|t). \tag{5.2}$$

The p-derivatives of the functions $f_j(z|s)$ obey the IDE

$$\begin{aligned} \partial_p f_j(z|s) = &\left(-\mathrm{i}k(z) f_j(z|s) + \frac{1}{2}\sum_{l=1}^{4} \left[B^{(0)}_{jl} + \frac{1}{z} B^{(1)}_{jl} \right] * f_l \Big|_{z,s} \right) (\delta_{j,3}+\delta_{j,4}) \\ &- \frac{1}{2z} \sum_{k=3}^{4} C^{(1)}_{jk} * \sum_{l=1}^{4} [I - \mathrm{i}B^{(1)}]_{kl} * f_l \Big|_{z,s} - \frac{1}{2}\sum_{k=3}^{4} C^{(0)}_{jk} * f_k \Big|_{z,s} \end{aligned} \tag{5.3}$$

where $I_{jk}(s,t) = \delta_{jk}\delta(s-t)$ and where the integral operators $B^{(n)}$ and $C^{(n)}$ are defined as

$$\begin{aligned} B^{(n)}_{jk}(s,t) &= \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \frac{\mathrm{d}z}{z^n} e_j(z|s) F_k(z|t) \\ C^{(n)}_{jk}(s,t) &= \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \frac{\mathrm{d}z}{z^n} f_j(z|s) E_k(z|t). \end{aligned} \tag{5.4}$$

We note the following relations between the integral operators $B^{(n)}_{jk}$ and $C^{(n)}_{jk}$

$$B^{(0)}_{jk}(s,t) = C^{(0)}_{jk}(s,t) \qquad [I - \mathrm{i}B^{(1)}]_{jk} * [I + \mathrm{i}C^{(1)}]_{kl}|_{s,t} = \delta_{jl}\delta(s-t). \tag{5.5}$$

These identities can be easily proved by using (4.7). From now on we will replace $B^{(0)}_{jk}$ in all expressions by $C^{(0)}_{jk}$. The IDE for $F_j(z|s)$ are found to be

$$\begin{aligned} \partial_p F_j(z|s) = &\left(\mathrm{i}k(z) F_j(z|s) + \frac{1}{2}\sum_{l=1}^{4} F_l * \left[C^{(0)}_{lj} + \frac{1}{z} C^{(1)}_{lj} \right] \Big|_{z,s} \right) (\delta_{j,3}+\delta_{j,4}) \\ &- \frac{1}{2z} \sum_{k=3}^{4}\sum_{l=1}^{4} F_l * [I + \mathrm{i}C^{(1)}]_{lk} * B^{(1)}_{kj} \Big|_{z,s} - \frac{1}{2}\sum_{k=3}^{4} F_k * C^{(0)}_{kj} \Big|_{z,s}. \end{aligned} \tag{5.6}$$

The 'potentials' $B^{(n)}$ and $C^{(n)}$ obey the equations

$$\begin{aligned} \partial_p C^{(n)}_{jk}(s,t) = &-\tfrac{1}{2} \sum_{m=3}^{4} C^{(0)}_{jm} * C^{(n)}_{mk} \Big|_{s,t} - \tfrac{1}{2} \sum_{m=3}^{4} C^{(1)}_{jm} * \sum_{l=1}^{4} [I - \mathrm{i}B^{(1)}]_{ml} * C^{(n+1)}_{lk} \Big|_{s,t} \\ &- \frac{\mathrm{i}}{2}(\delta_{j,3}+\delta_{j,4}) \Big[C^{(n-1)}_{jk}(s,t) - C^{(n+1)}_{jk}(s,t) \end{aligned}$$

$$+\mathrm{i}\sum_{l=1}^{4} C_{jl}^{(0)} * C_{lk}^{(n)}\Big|_{s,t} + B_{jl}^{(1)} * C_{lk}^{(n+1)}\Big|_{s,t}\Big]$$

$$+\frac{\mathrm{i}}{2}(\delta_{k,3}+\delta_{k,4})[C_{jk}^{(n-1)}(s,t) - C_{jk}^{(n+1)}(s,t)] \tag{5.7}$$

$$\partial_p B_{jk}^{(n)}(s,t) = -\tfrac{1}{2}\sum_{m=3}^{4} B_{jm}^{(n)} * C_{mk}^{(0)}\Big|_{s,t} - \tfrac{1}{2}\sum_{m=3}^{4}\sum_{l=1}^{4} B_{jl}^{(n+1)} * [I + \mathrm{i}C^{(1)}]_{lm} * B_{mk}^{(1)}\Big|_{s,t}$$

$$+\frac{\mathrm{i}}{2}(\delta_{k,3}+\delta_{k,4})\Big[B_{jk}^{(n-1)}(s,t) - B_{jk}^{(n+1)}(s,t)$$

$$-\mathrm{i}\sum_{l=1}^{4} B_{jl}^{(n)} * C_{lk}^{(0)}\Big|_{s,t} + B_{jl}^{(n+1)} * C_{lk}^{(1)}\Big|_{s,t}\Big]$$

$$-\frac{\mathrm{i}}{2}(\delta_{j,3}+\delta_{j,4})[B_{jk}^{(n-1)}(s,t) - B_{jk}^{(n+1)}(s,t)]. \tag{5.8}$$

The derivatives with respect to Λ are given by

$$\begin{aligned}
\partial_\Lambda C_{jk}^{(0)}(s,t) &= 2\mathrm{e}^{2\Lambda} f_j(\mathrm{e}^{2\Lambda}|s)F_k(\mathrm{e}^{2\Lambda}|t) + 2\mathrm{e}^{-2\Lambda} f_j(\mathrm{e}^{-2\Lambda}|s)F_k(\mathrm{e}^{-2\Lambda}|t)\\
\partial_\Lambda C_{jk}^{(1)}(s,t) &= 2f_j(\mathrm{e}^{2\Lambda}|s)(F_k(\mathrm{e}^{2\Lambda}|t) + \mathrm{i}F_l * C_{lk}^{(1)}|_{\mathrm{e}^{2\Lambda},t}) + \Lambda\to-\Lambda\\
\partial_\Lambda C_{jk}^{(2)}(s,t) &= 2\mathrm{e}^{-2\Lambda} f_j(\mathrm{e}^{2\Lambda}|s)(F_k(\mathrm{e}^{2\Lambda}|t) + \mathrm{i}F_l * C_{lk}^{(1)}|_{\mathrm{e}^{2\Lambda},t} + \mathrm{i}\mathrm{e}^{2\Lambda}F_l * C_{lk}^{(2)}|_{\mathrm{e}^{2\Lambda},t})\\
&\quad +\Lambda\to-\Lambda\\
\partial_\Lambda B_{jk}^{(1)}(s,t) &= 2F_k(\mathrm{e}^{2\Lambda}|t)(f_j(\mathrm{e}^{2\Lambda}|s) - \mathrm{i}B_{jl}^{(1)} * f_l|_{\mathrm{e}^{2\Lambda},s}) + \Lambda\to-\Lambda.
\end{aligned} \tag{5.9}$$

Equations (5.1), (5.3) and (5.6) form a Lax pair. Their consistency is implied by the following relation for the cross-derivatives

$$\partial_p\partial_\Lambda f_j(z|s) = \partial_\Lambda\partial_p f_j(z|s). \tag{5.10}$$

In order to simplify the computations we first introduce some notation. We rewrite (5.3) as

$$\partial_p f_j(z|s) = -\frac{\mathrm{i}}{2}zf_j(z|s)(\delta_{j,3}+\delta_{j,4}) + \sum_{l=1}^{4} a_{jl} * f_l\Big|_{z,s} + \frac{1}{z}\sum_{l=1}^{4} b_{jl} * f_l\Big|_{z,s} \tag{5.11}$$

where

$$\begin{aligned}
a_{jl}(s,t) &= \tfrac{1}{2}C_{jl}^{(0)}(\delta_{j,3}+\delta_{j,4}-\delta_{l,3}-\delta_{l,4})\\
b_{jl}(s,t) &= \frac{\mathrm{i}}{2}\delta_{jl}\delta(s-t) + \frac{1}{2}B_{jl}^{(1)}(\delta_{j,3}+\delta_{j,4}) - \frac{1}{2}\sum_{k=3}^{4} C_{jk}^{(1)}[I - \mathrm{i}B^{(1)}]_{kl}.
\end{aligned} \tag{5.12}$$

In the same notation (5.6) can be written as

$$\partial_p F_j(z|s) = \frac{\mathrm{i}}{2}zF_j(z|s)(\delta_{j,3}+\delta_{j,4}) - \sum_{l=1}^{4} F_l * a_{lj}\Big|_{z,s} - \frac{1}{z}\sum_{l=1}^{4} F_l * b_{lj}\Big|_{z,s}. \tag{5.13}$$

Similarly we introduce the notation

$$U_{jl}(z|s,t) = \frac{A_{jl}(\Lambda|s,t)}{z-\mathrm{e}^{2\Lambda}} + \frac{A_{jl}(-\Lambda|s,t)}{z-\mathrm{e}^{-2\Lambda}} \tag{5.14}$$

where $A_{jk}(\Lambda|s,t) = 2\mathrm{i}\mathrm{e}^{2\Lambda}\ f_j(\mathrm{e}^{2\Lambda}|s)F_k(\mathrm{e}^{2\Lambda}|t)$. In what follows we will denote by $\hat{A}_{jk}(\Lambda)$ the integral operator in the s-variable with kernel $A(\Lambda|s,t)$. After some calculations we arrive at the following equations

$$\partial_\Lambda \partial_p f_j(z|s) = \frac{\mathrm{i}z}{2}(\delta_{j,3}+\delta_{j,4})\left(\sum_{m=1}^{4}\frac{\hat{A}(\Lambda)_{jm} * f_m|_{z,s}}{z-\mathrm{e}^{2\Lambda}} + \Lambda \to -\Lambda\right) + \sum_{m=1}^{4}\partial_\Lambda a_{jm} * f_m\Big|_{z,s} - \sum_{l=1}^{4} a_{jl} * \left(\sum_{m=1}^{4}\frac{\hat{A}_{lm}(\Lambda) * f_m}{z-\mathrm{e}^{2\Lambda}} + \Lambda \to -\Lambda\right)\Big|_{z,s} + \frac{1}{z}\sum_{m=1}^{4}\partial_\Lambda b_{jm} * f_m\Big|_{z,s} - \frac{1}{z}\sum_{l=1}^{4} b_{jl} * \left(\sum_{m=1}^{4}\frac{\hat{A}_{lm}(\Lambda) * f_m}{z-\mathrm{e}^{2\Lambda}} + \Lambda \to -\Lambda\right)\Big|_{z,s} \tag{5.15}$$

$$\partial_p \partial_\Lambda f_j(z|s) = -\sum_{m=1}^{4}\left(\frac{\partial_p \hat{A}(\Lambda)_{jm} * f_m|_{z,s}}{z-\mathrm{e}^{2\Lambda}} + \Lambda \to -\Lambda\right) + \frac{\mathrm{i}z}{2}\sum_{m=1}^{4}\left(\frac{\hat{A}(\Lambda)_{jm} * f_m|_{z,s}}{z-\mathrm{e}^{2\Lambda}} + \Lambda \to -\Lambda\right)(\delta_{m,3}+\delta_{m,4}) - \sum_{l=1}^{4}\sum_{m=1}^{4}\left(\frac{\hat{A}(\Lambda)_{jl}}{z-\mathrm{e}^{2\Lambda}} + \Lambda \to -\Lambda\right) * \left(a_{lm} * f_m + \frac{1}{z} b_{lm} * f_m\right)\Big|_{z,s}. \tag{5.16}$$

In order to equate (5.15) and (5.16) we first rewrite both equations in the form $O_{jm} * f_m$, where O are complicated integral operators, and then 'truncate' the f_m's from the resulting expressions, which amounts to supposing that they form an independent set of functions in the space the integral operators act in. In the next step we then compare the resulting expressions (which are both meromorphic functions of z) at the singular points $z = \infty, 0, \mathrm{e}^{\pm 2\Lambda}$. If they (their residues) are equal at these points the expressions coincide for all values of z. For $z \to \infty$ we get the condition

$$\partial_\Lambda a_{jm}(s,t) = \frac{\mathrm{i}}{2}(\delta_{m,3}+\delta_{m,4}-\delta_{j,3}-\delta_{j,4})(A_{jm}(\Lambda|s,t) + A_{jm}(-\Lambda|s,t)). \tag{5.17}$$

At $z = 0$ we obtain

$$\partial_\Lambda b_{jm}(s,t) = \sum_{l=1}^{4}(\mathrm{e}^{-2\Lambda}[\hat{A}_{jl}(\Lambda) * b_{lm} - b_{jl} * \hat{A}_{lm}(\Lambda)] + \Lambda \to -\Lambda)\Big|_{s,t}. \tag{5.18}$$

At $z = \mathrm{e}^{2\Lambda}$ we obtain

$$\partial_p A_{jm}(\Lambda|s,t) = \frac{\mathrm{i}}{2}\mathrm{e}^{2\Lambda}(\delta_{m,3}+\delta_{m,4}-\delta_{j,3}-\delta_{j,4})A_{jm}(\Lambda|s,t) + \sum_{l=1}^{4}([a_{jl}+\mathrm{e}^{-2\Lambda}b_{jl}] * \hat{A}_{lm}(\Lambda) - \hat{A}_{jl}(\Lambda) * [a_{lm}+\mathrm{e}^{-2\Lambda}b_{lm}])\Big|_{s,t} \tag{5.19}$$

whereas the condition from $z = \mathrm{e}^{-2\Lambda}$ is obtained by taking $\Lambda \to -\Lambda$ in (5.19). It is straightforward to show that these equations hold by inserting the expressions for a, b and A and using the identities for the p- and Λ-derivatives of $C_{jk}^{(n)}$ and $B_{jk}^{(n)}$ written above.

Finally, to relate the functional determinant in (3.24) to the quantities introduced above we turn to the logarithmic derivatives of $\det(1 + \frac{1}{2\pi}\hat{V}^T)$.

The derivative with respect to p is given by

$$\partial_p \ln\left(\det\left(1 + \frac{1}{2\pi}\hat{V}^T\right)\right) = \mathrm{tr}\left((1-\hat{R}) * \frac{1}{2\pi}\partial_p \hat{V}^T\right). \tag{5.20}$$

Using (4.4) we find that

$$\frac{1}{2\pi}\partial_p V^T(z_1,z_2) = \frac{k(z_1)-k(z_2)}{z_1-z_2}\int_0^\infty \mathrm{d}s \sum_{j=3}^4 e_j(z_1|s)E_j(z_2|s)$$
$$= \frac{1}{2}\left(1+\frac{1}{z_1z_2}\right)\int_0^\infty \mathrm{d}s \sum_{j=3}^4 e_j(z_1|s)E_j(z_2|s). \tag{5.21}$$

This implies that

$$(1-\hat{R})*\frac{1}{2\pi}\partial_p\hat{V}^T\bigg|_{z_1,z_2} = \frac{1}{2}\int_0^\infty \mathrm{d}s\sum_{j=3}^4\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)}\mathrm{d}z\,[\delta(z_1-z)-R(z_1,z)]\frac{e_j(z|s)E_j(z_2|s)}{zz_2}$$
$$+\tfrac{1}{2}\int_0^\infty \mathrm{d}s\sum_{j=3}^4 f_j(z_1|s)E_j(z_2|s). \tag{5.22}$$

Using the representation (4.8) of $R(z_1,z_2)$ we rewrite the r.h.s. as

$$\text{r.h.s.} = \frac{1}{2}\int_0^\infty \mathrm{d}s\sum_{k=3}^4\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)}\mathrm{d}z\,[\delta(z_1-z)-R(z_1,z)]\frac{\mathrm{e}_k(z|s)E_k(z_2|s)}{z_1z_2}$$
$$-\frac{\mathrm{i}}{2}\int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}t\sum_{k=3}^4\sum_{l=1}^4\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)}\mathrm{d}z\,\frac{f_l(z_1|t)F_l(z|t)}{z_1z}\,\frac{\mathrm{e}_k(z|s)E_k(z_2|s)}{z_2}$$
$$+\tfrac{1}{2}\int_0^\infty \mathrm{d}s\sum_{j=3}^4 f_j(z_1|s)E_j(z_2|s). \tag{5.23}$$

Using this with (5.4) in (5.20) we finally arrive at

$$\partial_p \ln\left(\det\left(1+\frac{1}{2\pi}\hat{V}^T\right)\right) = \frac{1}{2}\sum_{k=3}^4\int_0^\infty \mathrm{d}s\left[C_{kk}^{(0)}(s,s)+C_{kk}^{(2)}(s,s)-\mathrm{i}\sum_{l=1}^4 B_{kl}^{(1)}*C_{lk}^{(2)}\bigg|_{s,s}\right]. \tag{5.24}$$

The logarithmic derivative of the determinant with respect to Λ is

$$\partial_\Lambda \ln\left(\det\left(1+\frac{1}{2\pi}\hat{V}^T\right)\right) = 2\mathrm{e}^{2\Lambda}R(\mathrm{e}^{2\Lambda},\mathrm{e}^{2\Lambda})+2\mathrm{e}^{-2\Lambda}R(\mathrm{e}^{-2\Lambda},\mathrm{e}^{-2\Lambda}). \tag{5.25}$$

After some manipulations similar to the case of the Bose gas (see [9]) this can be rewritten as

$$
\begin{aligned}
R(\mathrm{e}^{2\Lambda}, \mathrm{e}^{2\Lambda}) &= \frac{\mathrm{i}\mathrm{e}^{-2\Lambda}}{2} \sum_{j=1}^{4} \int_0^\infty \mathrm{d}s\, F_j(\mathrm{e}^{2\Lambda}|s) \frac{\mathrm{d}}{\mathrm{d}\Lambda} f_j(\mathrm{e}^{2\Lambda}|s) \\
&\quad - \frac{\mathrm{e}^{-4\Lambda}}{2\sinh(2\Lambda)} \left[\sum_{j=1}^{4} \int_0^\infty \mathrm{d}s\, f_j(\mathrm{e}^{-2\Lambda}|s) F_j(\mathrm{e}^{2\Lambda}|s) \right] \\
&\quad \times \left[\sum_{l=1}^{4} \int_0^\infty \mathrm{d}t\, f_l(\mathrm{e}^{2\Lambda}|t) F_l(\mathrm{e}^{-2\Lambda}|t) \right] \\
R(\mathrm{e}^{-2\Lambda}, \mathrm{e}^{-2\Lambda}) &= -\frac{\mathrm{i}\mathrm{e}^{2\Lambda}}{2} \sum_{j=1}^{4} \int_0^\infty \mathrm{d}s\, F_j(\mathrm{e}^{-2\Lambda}|s) \frac{\mathrm{d}}{\mathrm{d}\Lambda} f_j(\mathrm{e}^{-2\Lambda}|s) \\
&\quad - \frac{\mathrm{e}^{4\Lambda}}{2\sinh(2\Lambda)} \left[\sum_{j=1}^{4} \int_0^\infty \mathrm{d}s\, f_j(\mathrm{e}^{2\Lambda}|s) F_j(\mathrm{e}^{-2\Lambda}|s) \right] \\
&\quad \times \left[\sum_{l=1}^{4} \int_0^\infty \mathrm{d}t\, f_l(\mathrm{e}^{-2\Lambda}|t) F_l(\mathrm{e}^{2\Lambda}|t) \right].
\end{aligned}
\tag{5.26}
$$

This embeds the determinant into the system of integrable integro-differential equations derived above.

6. The Riemann–Hilbert problem

In this section we show that the results of the previous section can be reformulated in terms of an infinite-dimensional RHP for an integral operator valued function $Y(z)$. This connection will enable us to determine the asymptotic behaviour of the correlation function. We introduce the conjugation matrix $G(z)$ of this RHP as

$$[G(z|s,t)]_{ij} = \delta_{ij}\delta(s-t) + 2\pi\,\mathrm{e}_i(z|s)E_j(z|t). \tag{6.1}$$

It's elements can be expressed in terms of the projectors (4.5), e.g.

$$
\begin{aligned}
&[G(z|s,t)]_{11} = \delta(s-t) - \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))|2, z, s\rangle\langle 2, z, t| \\
&[G(z|s,t)]_{12} = \kappa \exp(\varphi_4(z))|2, z, s\rangle\langle 1, z, t| \\
&\ldots
\end{aligned}
$$

Consider now an integral-operator valued function $Y(z)$ with kernel $Y_{jk}(z|s,t)$, $j,k = 1,\ldots,4$, $s,t \in [0,\infty)$ acting on a vector f of functions of z and s according to

$$[Y(z) * f(z)]_j = \int_0^\infty \mathrm{d}t \sum_{k=1}^{4} Y_{jk}(z|s,t) f_k(z|t). \tag{6.2}$$

$Y(z)$ is solution to the following RHP

- $Y(z) = I + \sum_{k=1}^{\infty} \frac{M_k}{z^k}$ for $z \to \infty$.
- $Y(z)$ is analytic throughout the complex plane with the exception of the contour C, which is the interval $[\exp(-2\Lambda), \exp(2\Lambda)]$ on the real axis (see figure 1).
- $Y^-(z) = Y^+(z)G(z)$ on C where $Y^\pm(z)$ are the boundary values as indicated in figure 1 and $G(z)$ is the conjugation matrix (6.1).

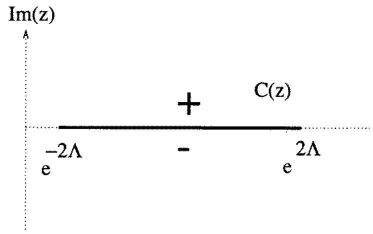

Figure 1. Conjugation contour for the RHP.

This RHP can be rewitten as the system of singular integral equations

$$Y^{+}(z) = I + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}z' \, \frac{Y^{+}(z')[I - G(z')]}{z' - z - \mathrm{i}0}. \tag{6.3}$$

The solution of (6.3) can be expressed in terms of the functions E and f defined in section 4 as

$$Y^{+}_{ij}(z|s,t) = \delta_{ij}\delta(s-t) + \mathrm{i} \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \mathrm{d}z' \, \frac{f_i(z'|s)E_j(z'|t)}{z' - z - \mathrm{i}0} \tag{6.4}$$

which follows from the identity

$$f_j(z|s) = \int_0^{\infty} \mathrm{d}t \; Y_{jk}(z|s,t) e_j(z|t). \tag{6.5}$$

The potentials $B^{(1)}$ and $C^{(n)}$ (5.4) can be related to the solution $Y(z)$ of the RHP through asymptotic expansions around 0 and ∞. We find

$$Y_{jk}(z) \longrightarrow I + \mathrm{i}C^{(1)} + \mathrm{i}zC^{(2)} + \mathrm{i}z^2C^{(3)} + \mathcal{O}(z^3) \qquad \text{for } z \to 0 \tag{6.6}$$

$$Y_{jk}(z) \longrightarrow I - \frac{\mathrm{i}}{z}C^{(0)} + \mathcal{O}(z^{-2}) \qquad \text{for } z \to \infty. \tag{6.7}$$

From (6.6) and (5.5) we find

$$[I - \mathrm{i}B^{(1)}] * C^{(2)} = -\mathrm{i}(Y^{-1}(0) \frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} Y(z)). \tag{6.8}$$

Together with (5.24) this expresses the correlation function (3.23) in terms of the solution $Y(z)$ of our RHP.

6.1. Analysis of the RHP

While the operator-valued RHP defined above determines the correlation functions completely, its solution appears to be a daunting task in general. In what follows we concentrate on the leading term in the asymptotical decomposition of the solution of the RHP in the region of coupling constant $\frac{\pi}{2} < \gamma < \frac{2\pi}{3}$. The reason for this restriction is the following: the upper bound on γ stems from the construction of the ground state of our lattice regularization. The lower bound ensures that the parameter p defined in (4.2) will go to infinity in the continuum limit, which essentially simplifies the analysis of the RHP: it permits us to study the asymptotical decomposition of the solution of the RHP with respect to p (recall that p contains the continuum distance as well). Due to the fact that this parameter will be not only large but diverge the number of terms in the asymptotical decomposition will be very small—in fact we expect only three contributions (see also below). As we shall show in our analysis of the leading contribution, the special form of the conjugation matrix in addition to our interest in partial traces of Y only allows to reduce the RHP to a tractable scalar one (still containing the auxiliary dual fields, of course). The analysis of the subleading terms is technically much more involved and is currently under investigation. We will report on this work elsewhere.

Let us now turn to the calculation of the leading term. First, we note that the conjugation matrix can be decomposed into the product of an upper and lower triagonal matrix as follows

$$[G(z|s,t)]_{ab} = \sum_{c=1}^{4} \int_0^\infty \mathrm{d}s' \, [T_1(z|s,s')]_{ac}[T_2(z|s',t)]_{cb}. \tag{6.9}$$

Here

$$T_1(z|s,t) = \begin{pmatrix} 1 & \alpha_1(z|s,t) & \alpha_2(z|s,t)\exp(\mathrm{i}pk(z)) & \alpha_3(z|s,t)\exp(\mathrm{i}pk(z)) \\ 0 & 1 & \alpha_4(z|s,t)\exp(\mathrm{i}pk(z)) & \alpha_5(z|s,t)\exp(\mathrm{i}pk(z)) \\ 0 & 0 & 1 & \alpha_6(z|s,t) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.10}$$

with matrix elements

$$\begin{aligned}
\alpha_1(z|s,t) &= \frac{\kappa \exp(\varphi_4(z))}{1+\kappa^2\exp(\varphi_4(z)-\varphi_3(z))} |2,z,s\rangle\langle 1,z,t| \\
\alpha_2(z|s,t) &= -\frac{1}{\kappa} \exp(-\varphi_2(z)+\varphi_3(z))|2,z,s\rangle\langle 2,z,t| \\
\alpha_3(z|s,t) &= \frac{\kappa^2 \exp(\varphi_1(z)-\varphi_3(z)+\varphi_4(z))}{1+\kappa^2\exp(\varphi_4(z)-\varphi_3(z))} |2,z,s\rangle\langle 1,z,t| \\
\alpha_4(z|s,t) &= -\frac{1}{\kappa^2} \exp(-\varphi_2(z)+\varphi_3(z)-\varphi_4(z))|1,z,s\rangle\langle 2,z,t| \\
\alpha_5(z|s,t) &= \frac{\kappa \exp(\varphi_1(z)-\varphi_3(z))}{1+\kappa^2\exp(\varphi_4(z)-\varphi_3(z))} |1,z,s\rangle\langle 1,z,t| \\
\alpha_6(z|s,t) &= \frac{\kappa \exp(\varphi_2(z)+\varphi_1(z)-\varphi_3(z))}{1+\kappa^2\exp(\varphi_4(z)-\varphi_3(z))} |2,z,s\rangle\langle 1,z,t|.
\end{aligned} \tag{6.11}$$

Similarly, we find

$$T_2(z|s,t) = \begin{pmatrix} c_1(z|s,t) & 0 & 0 & 0 \\ \beta_1(z|s,t) & c_2(z|s,t) & 0 & 0 \\ \beta_2(z|s,t)\exp(-\mathrm{i}pk(z)) & \beta_4(z|s,t)\exp(-\mathrm{i}pk(z)) & c_3(z|s,t) & 0 \\ \beta_3(z|s,t)\exp(-\mathrm{i}pk(z)) & \beta_5(z|s,t)\exp(-\mathrm{i}pk(z)) & \beta_6(z|s,t) & c_4(z|s,t) \end{pmatrix}. \tag{6.12}$$

The matrix elements of T_2 are given by

$$\begin{aligned}
\beta_1(z|s,t) &= -\frac{1}{\kappa} \exp(-\varphi_4(z))|1,z,s\rangle\langle 2,z,t| \\
\beta_2(z|s,t) &= -\frac{\kappa \exp(\varphi_2(z)-\varphi_3(z))}{1+\kappa^2\exp(\varphi_4(z)-\varphi_3(z))} |2,z,s\rangle\langle 2,z,t| \\
\beta_3(z|s,t) &= -\kappa^2 \exp(-\varphi_1(z)-\varphi_3(z)+\varphi_4(z))|1,z,s\rangle\langle 2,z,t| \\
\beta_4(z|s,t) &= \frac{\exp(\varphi_2(z))}{1+\kappa^2\exp(\varphi_4(z)-\varphi_3(z))} |2,z,s\rangle\langle 1,z,t| \\
\beta_5(z|s,t) &= \kappa \exp(-\varphi_1(z)+\varphi_4(z))|1,z,s\rangle\langle 1,z,t| \\
\beta_6(z|s,t) &= -\kappa \exp(-\varphi_1(z)-\varphi_2(z)+\varphi_4(z))|1,z,s\rangle\langle 2,z,t|
\end{aligned} \tag{6.13}$$

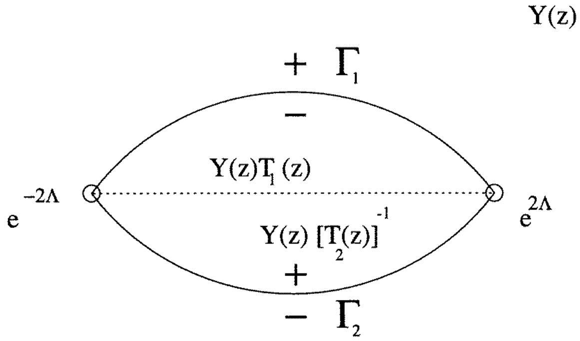

Figure 2. Deformation of the conjugation contour for the RHP.

and finally

$$\begin{aligned} c_1(z|s,t) &= 1 - \frac{\kappa^2 \exp(\varphi_4(z) - \phi_3(z))}{1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))} |2\rangle\langle 2| \\ c_2(z|s,t) &= 1 + \frac{1}{\kappa^2} \exp(-\varphi_4(z) + \varphi_3(z)) |1\rangle\langle 1| \\ c_3(z|s,t) &= 1 - \frac{1}{1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))} |2\rangle\langle 2| \\ c_4(z|s,t) &= 1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z)) |1\rangle\langle 1|. \end{aligned} \tag{6.14}$$

Let us now go through a 'deformation' of the RHP like for the case of the Bose gas [28]. We define an integral-operator valued function $\tilde{Y}(z)$ in the following way:

• $\tilde{Y}(z) = Y(z)$ outside the 'bubble' defined in figure 2. In particular $\tilde{Y}(z) = Y(z)$ for $z \to 0, \infty$, which will be important later.

• $\tilde{Y}(z) = Y(z)T_1(z)$ in the region enclosed by the real axis and the contour Γ_1. Note that in this region $\operatorname{Im} k(z) \geqslant 0 \ \forall z$.

• $\tilde{Y}(z) = Y(z)[T_2(z)]^{-1}$ in the region enclosed by the real axis and the contour Γ_2. Note that in this region $\operatorname{Im} k(z) \leqslant 0 \ \forall z$.

It can be easily seen that the function $Y(z)$ defined in the above way has the following properties: $\tilde{Y}(z)$ is analytic in the whole complex plane with the exception of the contours Γ_1 and Γ_2. On the contours Γ_j $\tilde{Y}$ satisfies the conjugation equations

$$\begin{aligned} (\tilde{Y})^-(z) &= \tilde{Y}^+(z)T_1(z) \qquad z \in \Gamma_1 \\ (\tilde{Y})^-(z) &= \tilde{Y}^+(z)T_2(z) \qquad z \in \Gamma_2. \end{aligned} \tag{6.15}$$

Since we are only interested in the asymptotic behaviour of the determinant for $p \gg 1$ we can use the fact that in this limit $T_{1(2)}$ become blockdiagonal in the vicinity of the contour $\Gamma_{1(2)}$ from which we find that

$$Y(z) \sim \begin{pmatrix} \tilde{\Phi}_1(z) & 0 \\ 0 & \tilde{\Phi}_2(z) \end{pmatrix}. \tag{6.16}$$

Here $\tilde{\Phi}_j(z)$ are solutions to 2×2 operator-valued RHPs

$$\tilde{\Phi}_j^-(z) = \tilde{\Phi}_j^+(z) * G_j(z) \qquad j = 1, 2 \tag{6.17}$$

with the same conjugation contour C as the original RHP and conjugation matrices

$$G_1(z) = \begin{pmatrix} 1-|2\rangle\langle 2| & 0 \\ 0 & 1-|1\rangle\langle 1| \end{pmatrix} + \begin{pmatrix} 0 & \frac{\exp(\varphi_3(z))}{\kappa}|2\rangle\langle 1| \\ -\frac{\exp(-\varphi_4(z))}{\kappa}|1\rangle\langle 2| & (1+\frac{\exp(\varphi_3(z)-\varphi_4(z))}{\kappa^2})|1\rangle\langle 1| \end{pmatrix}$$

$$G_2(z) = \begin{pmatrix} 1-|2\rangle\langle 2| & 0 \\ 0 & 1-|1\rangle\langle 1| \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & \kappa\exp(\varphi_1(z)+\varphi_2(z)-\varphi_3(z))|2\rangle\langle 1| \\ -\kappa\exp(-\varphi_1(z)-\varphi_2(z)+\varphi_4(z))|1\rangle\langle 2| & (1+\kappa^2\exp(\varphi_4(z)-\varphi_3(z)))|1\rangle\langle 1| \end{pmatrix}. \tag{6.18}$$

Using the fact that $G_j(z)$ form representations of $GL(2|C)$ we can now calculate the determinants of $G_j(z)$ as is shown in the appendix

$$\det(G_1(z)) = \exp(-\alpha+\varphi_3(z)-\varphi_4(z)) \qquad \det(G_2(z)) = \exp(\alpha-\varphi_3(z)+\varphi_4(z)). \tag{6.19}$$

The scalar RHPs for the determinants

$$\det(\tilde{\Phi}_j^-(z)) = \det(\tilde{\Phi}_j^+(z))\det(G_j(z)) \qquad j=1,2$$

is now easily integrated to give

$$\det(\tilde{\Phi}_1(z)) = \exp\left(-\frac{1}{2\pi \mathrm{i}}\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \mathrm{d}z_1\, \frac{-\alpha+\varphi_3(z_1)-\varphi_4(z_1)}{z_1-z}\right)$$
$$\det(\tilde{\Phi}_2(z)) = \exp\left(-\frac{1}{2\pi \mathrm{i}}\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \mathrm{d}z_1\, \frac{\alpha-\varphi_3(z_1)+\varphi_4(z_1)}{z_1-z}\right). \tag{6.20}$$

7. Leading term in the asymptotics of the correlator

Let us now relate the solution of the scalar RHPs to the logarithmic derivative of $\det(1+\frac{1}{2\pi}\hat{V}^T)$. The contribution due to $C^{(0)}$ in (5.24) can be obtained from (6.7) and (6.20) as

$$\frac{1}{2}\sum_{k=3}^{4}\int_0^\infty \mathrm{d}s\, C_{kk}^{(0)}(s,s) = \lim_{z\to\infty}\frac{\mathrm{i}z}{2}\ln(\det(\tilde{\Phi}_2(z)))$$
$$= \frac{1}{4\pi}\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \mathrm{d}z\,[\alpha-\varphi_3(z)+\varphi_4(z)]. \tag{7.1}$$

Similarly, the second contribution in (5.24) is with (6.7)

$$\frac{1}{2}\sum_{k=3}^{4}\int_0^\infty \mathrm{d}s\,([I-\mathrm{i}B^{(1)}]*C^{(2)})_{kk}(s,s) = -\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0}\ln[\det(\tilde{\Phi}_2(z))]$$
$$= \frac{1}{4\pi}\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \mathrm{d}z\,\frac{\alpha-\varphi_3(z)+\varphi_4(z)}{z^2}. \tag{7.2}$$

Combining these to the leading asymptotical behaviour of $\partial_p\ln(\det(1+\frac{1}{2\pi}\hat{V}^T))$ and using the fact that they are p-independent we obtain

$$\det\left(1+\frac{1}{2\pi}\hat{V}^T\right) = A\exp\left(\frac{\alpha p\sinh(2\Lambda)}{\pi}\right)$$
$$\times\exp\left(\frac{p}{4\pi}\int_{\exp(-2\Lambda)}^{\exp(2\Lambda)}\mathrm{d}z\,\left(1+\frac{1}{z^2}\right)(\varphi_4(z)-\varphi_3(z))\right)$$

$$= A \exp\left(\frac{\alpha p \sinh(2\Lambda)}{\pi}\right) \exp\left(\frac{p}{\pi}\int_{-\Lambda}^{\Lambda} d\lambda \cosh(2\lambda)(\varphi_4(\lambda) - \varphi_3(\lambda))\right) \tag{7.3}$$

where A is a p-independent constant and where in the last step we have changed back to the original λ-variables. Decomposing the combination of dual fields into 'momenta' and 'coordinates' and using the commutation relations (3.19) we find

$$\varphi_4(\lambda) - \varphi_3(\lambda) = P(\lambda) + Q(\lambda) \qquad [Q(\mu), P(\lambda)] = 0. \tag{7.4}$$

This enables us to trivially evaluate the expectation value with respect to the dual fields in this approximation: the dual fields are found not to contribute at all leading to the following result for the leading asymptotical behaviour of the correlator

$$\langle\Omega|\exp(\alpha Q_1(n))|\Omega\rangle \sim \tilde{A} \exp\left(\frac{\alpha p}{\pi}\sinh(2\Lambda)\right) \tag{7.5}$$

where $\tilde{A}$ is a constant independent on p.

We will now argue that the approximation (7.5) is too crude due to the fact that we have neglected the influence of the dual fields in the *subleading* factors in the solution of the RHP. We expect the final answer for the solution of the RHP to be of the form

$$\det\left(1 + \frac{1}{2\pi}\hat{V}^T\right) = C(\{\varphi_j\}) \exp(\zeta(\{\varphi_j\})\ln(p)) \exp\left(\frac{\alpha p \sinh(2\Lambda)}{\pi}\right)$$
$$\times \exp\left(\frac{p}{\pi}\int_{-\Lambda}^{\Lambda} \mathrm{d}\lambda\ \cosh(2\lambda)(\varphi_4(\lambda) - \varphi_3(\lambda))\right) \tag{7.6}$$

where we keep in mind that $p \to \infty$ as the lattice spacing $\Delta \to 0$. In (7.6) C is p-independent and we have conjectured that the subleading term in the solution of the RHP is a power-law in p. Evaluating the expectation value of (7.6) in the dual bosonic Fock space *the dual fields will contribute in the exponential term,* i.e.

$$\langle\exp(\alpha Q_1(n))\rangle \sim (\tilde{0}|\det\left(1 + \frac{1}{2\pi}\hat{V}^T\right)|0)$$
$$= \tilde{C} p^{\tilde{\zeta}} \exp(\tilde{m} p) \exp(\xi p \ln(p)) \exp\left(\left[\frac{\sinh(2\Lambda)}{\pi} + \omega\right]\alpha p\right). \tag{7.7}$$

Here $\tilde{\zeta}$, ω and $\tilde{m}$ are functions of γ, the soliton mass etc. For this answer to be of the correct qualitative form, the following conditions have to be satisfied:

- $\xi = 0$, as the leading asymptotic behaviour should be exp(constant p).
- The last factor in (7.7) has to be cancelled by a suitable regularization procedure for the result to make sense. In the continuum limit we have (3.2) which implies that

$$\langle\exp(\alpha Q_1(n))\rangle \to \left\langle\exp\left(\frac{2\alpha}{\beta}[u(x) - u(0)]\right)\right\rangle \times \exp\left(\frac{\alpha x}{\Delta}\frac{\pi - \gamma}{2\gamma}\right). \tag{7.8}$$

We see that this expression contains a divergent factor depending both on α and on the distance x. We now adjust our 'cut-off' Λ in such a way that the divergent factor in (7.7) precisely reproduces the divergent factor in (7.8), i.e.

$$\exp\left(\frac{\alpha x}{\Delta}\frac{\pi - \gamma}{2\gamma}\right) = \exp\left(\left[\frac{\sinh(2\Lambda)}{\pi} + \omega\right]\alpha p\right).$$

If $\omega = 0$ this leads to the following relation between the 'cut-off' Λ and the lattice spacing Δ

$$\exp(2\Lambda) = \Delta^{-2\frac{\pi-\gamma}{\pi}}\left(\frac{8\pi(\pi - \gamma)}{c^2\sin(\gamma)\gamma}\right) \tag{7.9}$$

with a finite constant c. The procedure outlined above fixes the relation between Λ and Δ and thus between the divergent part of p and Λ. Note, however, that the result (7.9) for this relation is not consistent with the requirement $S\cosh(2\lambda) \ll 1$, which we have used in order to simplify the kernel of $\hat{V}$ in section 4. Therefore the assumption $\omega = 0$ has to be wrong and we do need a Λ-dependence of ω instead which corrects (7.9).

8. Summary and conclusion

In this paper we have applied the method of [9] to correlation fucntions in the sine–Gordon model. In order to deal with the ultraviolet divergences we used an integrable lattice regularization of the sine–Gordon model to derive a determinant representation for quantum correlation functions. We then took the continuum limit and obtained a determinant representation for the sine–Gordon QFT. Furthermore we embedded the determinant in a system of integrable integro-differential equations which we showed to be associated with an operator-valued RHP. The quantum correlation function was expressed in terms of the solution of this RHP. We then presented a general approach to obtain the leading asymptotical behaviour of the solution of the RHP, which in turn yields the leading term in the asymptotics of the quantum correlation function. We showed that the subleading terms in the asymptotical decomposition are essential for obtaining explicit expressions for the asymptotics of the correlation function due to the presence of the dual quantum fields. For the case at hand there appear to be only two subleading terms in the asymptotical decomposition which is very encouraging! The analysis of the subleading terms is a difficult mathematical problem by itself and we will report on it in a separate publication.

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Appendix. $Gl(2|C)$ Representation by integral operators

In this appendix we show how to essentially simplify the analysis of the operator-valued RHP through the use of $GL(2|C)$ representation theory. We closely follow the discussion of [28].

Let us consider an integral-operator valued 2×2 matrix with kernel

$$\mathcal{O}(s,t) = \begin{pmatrix} \mathcal{O}_{11}(s,t) & \mathcal{O}_{12}(s,t) \\ \mathcal{O}_{21}(s,t) & \mathcal{O}_{22}(s,t) \end{pmatrix} \qquad s,t \in [0,\infty). \tag{A.1}$$

Multiplication of integral-operator valued matrices $\mathcal{O}$ and $\mathcal{P}$ is defined in the usual way as

$$[\mathcal{O}\mathcal{P}]_{ij}(s,t) = \sum_{k=1}^{2} \int_0^\infty \mathrm{d}r\, \mathcal{O}_{ik}(s,r)\mathcal{P}_{kj}(r,t) \qquad i,j = 1,2. \tag{A.2}$$

The left (right) action of the integral operators $\mathcal{O}_{ij}$ on fuctions defined on the interval $[0,\infty)$ is given by

$$\mathcal{O}_{ij} * f|_s = \int_0^\infty \mathrm{d}t\, \mathcal{O}_{ij}(s,t) f(t) \qquad g * \mathcal{O}_{ij}|_t = \int_0^\infty \mathrm{d}s\, f(s)\mathcal{O}_{ij}(s,t). \tag{A.3}$$

Let us now construct a special class of such operators $\hat{\mathcal{O}}$ which form a representation of $Gl(2,\mathbb{C})$: we start with two pairs of functions $(\alpha(s),\beta(s))$ and $(A(s),B(s))$ on $[0,\infty)$ which we represent in Dirac notation as $\alpha(s)\equiv|1\rangle$, $\beta(s)\equiv|2\rangle$, $A(s)\equiv\langle 1|$ and $B(s)\equiv\langle 2|$. These functions are chosen in such a way that

$$\langle 1|1\rangle \equiv \int_0^\infty \mathrm{d}s\, A(s)\alpha(s) = 1 = \langle 2|2\rangle \equiv \int_0^\infty \mathrm{d}s\, B(s)\beta(s). \tag{A.4}$$

In this notation we may write left multiplication by $\hat{\mathcal{O}}_{ik}$ as

$$\hat{\mathcal{O}}_{ik}|1\rangle = \int_0^\infty \mathrm{d}t\, \mathcal{O}_{ik}(s,t)\alpha(t). \tag{A.5}$$

Observe now that one may define a representation $\hat{\mathcal{A}}$ of $Gl(2,\mathbb{C})$ in terms of integral operators via

$$M\in Gl(2,\mathbb{C}) \longmapsto \hat{\mathcal{A}}(M) = \begin{pmatrix} I-|1\rangle\langle 1| & 0 \\ 0 & I-|2\rangle\langle 2| \end{pmatrix} + \begin{pmatrix} M_{11}|1\rangle\langle 1| & M_{12}|1\rangle\langle 2| \\ M_{21}|2\rangle\langle 1| & M_{22}|2\rangle\langle 2| \end{pmatrix}. \tag{A.6}$$

Here M_{11}, M_{12}, M_{21} and M_{22} are complex numbers and I is the identity operator in the space of integral oprators on $[0,\infty)$. Multiplication by the integral operators $|1\rangle\langle 1|$, $|1\rangle\langle 2|$, $|2\rangle\langle 1|$ and $|2\rangle\langle 2|$ is given by e.g.

$$|1\rangle\langle 2|f(s) = \left(\int_0^\infty \mathrm{d}s\, B(s)f(s)\right)|1\rangle. \tag{A.7}$$

Therefore $|i\rangle\langle j|$ act like projectors on the 'states' $|i\rangle$ and $\langle j|$.

In particular identities like $[I-|1\rangle\langle 1|]|1\rangle\langle 1| = 0$ are seen to hold. Indeed for any M, $N\in Gl(2,\mathbb{C})$ the representation $\hat{\mathcal{A}}$ has the following properties

$$\text{(P1)}\qquad \hat{\mathcal{A}}(MN) = \hat{\mathcal{A}}(M)\hat{\mathcal{A}}(N) \qquad \hat{\mathcal{A}}(I) = I \qquad \hat{\mathcal{A}}(M^{-1}) = \hat{\mathcal{A}}^{-1}(M)$$

$$\text{(P2)}\qquad \mathrm{Tr}\left(\hat{\mathcal{A}}(M) - \begin{pmatrix} I-|1\rangle\langle 1| & 0 \\ 0 & I-|2\rangle\langle 2| \end{pmatrix}\right) = \mathrm{tr}\, M = M_{11}+M_{22} \tag{A.8}$$

$$\text{(P3)}\qquad \mathrm{Det}\,\hat{\mathcal{A}}(M) = \det M = M_{11}M_{22} - M_{12}M_{21}.$$

Properties (P1) and (P2) can be established by direct computation using the rules given above. Property (P3) shows that the determinant of the integral operator $\mathcal{A}$ is simply equal to the determinant of the 2×2 matrix M, which is quite remarkable. It is established by expressing the determinant as a trace *via* $\ln \mathrm{Det}\,\mathcal{A} = \mathrm{tr}\ln\mathcal{A}$, then using (P1) in the expansion of the logarithm, using (P2) to express the operator trace in terms of the matrix trace, and finally expressing the sum over traces back as determinant of the matrix M.

It can be easily checked that the representation (6.18) of the conjugation matrices $G_j(z)$ is precisely of the above form (here z plays the role of a parameter), which in turn allows us to evaluate the determinants of the conjugation matrices.

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